65. If $\sqrt{3}\sin\theta$ is = $\cos\theta$, find the value of $\dfrac{\sin\theta.\tan\theta.\ (1+\cot\theta)}{\sin\theta+\cos\theta}$

2015/2016 (1 Mark)

$\sqrt{3}\sin\theta = \cos\theta \rightarrow \frac{\sin\theta}{\cos\theta} = \frac{1}{\sqrt{3}}$.

$\rightarrow \tan\theta = \frac{1}{\sqrt{3}}$ ....(1)

Now we have: $\dfrac{\sin\theta\,.\tan\theta\,.(1+\cot\theta)}{\sin\theta+\cos\theta} = \dfrac{\frac{\sin\theta}{\cos\theta}.\tan\theta\,.(1+\cot\theta)}{\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\cos\theta}}$

(Dividing numerator and denominator by cos θ)

$= \dfrac{\tan\theta\tan\theta.(1+\cot\theta)}{\tan\theta+1} = \dfrac{\frac{1}{\sqrt{3}}\times.\frac{1}{\sqrt{3}}(1+\sqrt{3})}{\frac{1}{\sqrt{3}}+1} = \dfrac{\frac{1+\sqrt{3}}{3}}{\frac{1+\sqrt{3}}{\sqrt{3}}} = \dfrac{(1+\sqrt{3})\times\sqrt{3}}{3(1+\sqrt{3})}$.

$= \dfrac{\sqrt{3}}{3} = \dfrac{1}{\sqrt{3}}$.

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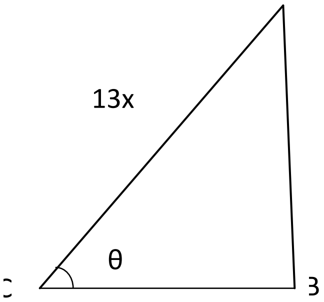
66. If $\sin\theta = \dfrac{12}{13}$, $0^\circ < \theta < 90^\circ$, find the value of $\dfrac{\sin^2\theta-\cos^2\theta}{2\sin\theta.\cos\theta}\times\dfrac{1}{\tan^2\theta}$.

2015/2016 (3 Mark)



$\sin\theta = \frac{12}{13}$.

So, if AB = 12x and AC =13x.

Then, $BC^2 = AC^2 - AB^2$

$=(13x)^2 - (12x)^2$

$= 169x^2 - 144x^2 = 25x^2$

$BC = \sqrt{25x^2} = 5x$

So, the value of $\dfrac{\sin^2\theta-\cos^2\theta}{2\sin\theta\,.\,\cos\theta}\times\dfrac{1}{\tan^2\theta}$.

$= \dfrac{\left(\frac{12x}{13x}\right)^2-\left(\frac{5x}{13x}\right)^2}{2\times\frac{12x}{13x}\times\frac{5x}{13x}}\times\dfrac{1}{\left(\frac{12x}{5x}\right)^2} = \dfrac{\left(\frac{144}{169}-\frac{25}{169}\right)}{2\times\frac{12}{13}\times\frac{5}{13}}\times\dfrac{25}{144}$

$= \dfrac{119}{169}\times\dfrac{169}{2\times12\times5}\times\dfrac{25}{144} = \dfrac{119\times5}{24\times144} = \dfrac{595}{3456}$.

---

67. If 3 tan A = 4, then prove that:

(i) $\sqrt{\frac{\sec A - \text{cosec } A}{\sec A + \text{cosec } A}} = \frac{1}{\sqrt{7}}$ (ii) $\sqrt{\frac{1-\sin A}{1+\cos A}} = \frac{1}{2\sqrt{2}}$

2015/2016 (4 Mark)

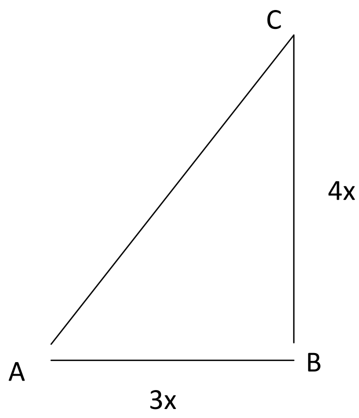
$3 \tan A = 4 \rightarrow \tan A = \frac{4}{3}$.

So, if BC = 4x, then AB=3x.

Hence, $AC^2 = AB^2 + BC^2$

$=(3x)^2 + (4x)^2 = 9x^2 + 16x^2 = 25x^2$

$\rightarrow$ $AC = \sqrt{25x^2} = 5x$.



So, we have: $\sec A = \frac{AC}{AB} = \frac{5x}{3x} = \frac{5}{3}$,

$\text{cosec } A = \frac{AC}{BC} = \frac{5x}{4x} = \frac{5}{4}$,

$\sin A = \frac{BC}{AC} = \frac{4x}{5x} = \frac{4}{5}$,

$\cos A = \frac{AB}{AC} = \frac{3x}{5x} = \frac{3}{5}$,

Thus, we have:

(i)L.H.S = $\sqrt{\frac{\sec A - \text{cosec}A}{\sec A + \text{cosec}A}} = \sqrt{\frac{\frac{5}{3} - \frac{5}{4}}{\frac{5}{3} + \frac{5}{4}}} = \sqrt{\frac{(20-15)\times 12}{12(20+15)}} = \sqrt{\frac{5}{35}} = \frac{1}{\sqrt{7}}$ =R.H.S

(ii)L.H.S = $\sqrt{\frac{1-\sin A}{1+\cos A}} = \sqrt{\frac{1-\frac{4}{5}}{1+\frac{3}{5}}} = \sqrt{\frac{(5-4)\times 5}{5(5+3)}} = \sqrt{\frac{1}{8}} = \frac{1}{2\sqrt{2}}$ =R.H.S

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68. If $\cos\theta = \frac{3}{5}$, find the value of $\left(\frac{5 \text{ cosec } \theta - 4 \tan\theta}{\sec\theta + \cot\theta}\right)$.

2015/2016 [4 marks]

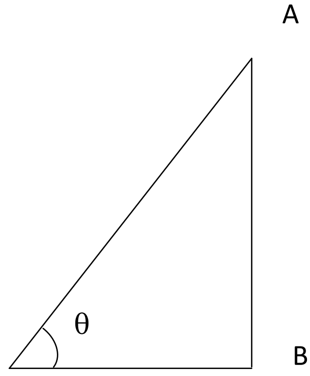
$\cos\theta = \frac{3}{5}$.

So, if BC = 3x, then AC = 5x.

Hence, $AB^2 = AC^2 - BC^2$

$=(5x)^2 - (3x)^2 = 16x^2$

$\rightarrow$ $AB = \sqrt{16x^2} = 4x$.



So, we have: $\text{cosec }\theta = \frac{AC}{AB} = \frac{5x}{4x} = \frac{5}{4}$

$\sec\theta = \frac{AC}{BC} = \frac{5x}{3x} = \frac{5}{3}$

$\tan\theta = \frac{AB}{BC} = \frac{4x}{3x} = \frac{4}{3}$

$\cot\theta = \frac{BC}{AB} = \frac{3x}{4x} = \frac{3}{4}$

Now, $\frac{5\ cosec\ \theta - 4\tan\theta}{\sec\theta + \cot\theta} = \frac{5\times\frac{5}{4} - 4\times\frac{4}{3}}{\frac{5}{3}+\frac{3}{4}} = \frac{\frac{25}{4}-\frac{16}{3}}{\frac{5}{3}+\frac{3}{4}} = \frac{\left(\frac{75-64}{12}\right)}{\frac{20+9}{12}} = \frac{11}{12}\times\frac{12}{29} = \frac{11}{29}$.

---

69. If cos (A + B) = $\frac{1}{2}$ = sin (A – B), then find A and B, when it is given that A+B and A – B are acute angles.

2014/2015/2016 [4 marks]

$\cos(A + B) = \frac{1}{2} \rightarrow A + B = 60^\circ$ .....(1)

$\sin(A - B) = \frac{1}{2} \rightarrow A - B = 30^\circ$ .....(2)

Adding eqns. (1) and (2),

$2A = 90^\circ \rightarrow A = \frac{90^\circ}{2} = 45^\circ$.

Putting $A = 45^\circ$ in eqn. (1) , we get

$45^\circ + B = 60^\circ \rightarrow B = 60^\circ - 45^\circ = 15^\circ$.

---

70. Find the value of $\tan(65^\circ - \theta) - \cot(25^\circ + \theta)$.

2014/2015/2016 [1 mark]

$\tan(65^\circ - \theta) - \cot(25+\theta)$

$= \tan(65^\circ - \theta) - \cot[90^\circ-(65^\circ-\theta)]$

$= \tan(65^\circ - \theta) - \tan(65^\circ - \theta) = 0$.

---

71. Find the value of $\frac{1}{3}\times\frac{\cos 36^\circ}{\sin 54^\circ} - \frac{3}{2}\times\frac{\sec 16^\circ}{cosec 74^\circ}$.

2014/2015/2016 [1 mark]

$\frac{1}{3}\cdot\frac{\cos 36^\circ}{\sin 54^\circ} - \frac{3}{2}\cdot\frac{\sec(16^\circ)}{cosec\ 74^\circ}$

$= \frac{1}{3}\times\frac{\cos(90^\circ-54^\circ)}{\sin 54^\circ} - \frac{3}{2}\times\frac{\sec(90^\circ-74^\circ)}{cosec\ 74^\circ}$

$= \frac{1}{3}\times\frac{\sin 54^\circ}{\sin 54^\circ} - \frac{3}{2}\times\frac{cosec\ 74^\circ}{cosec\ 74^\circ} = \frac{1}{3}\times 1 - \frac{3}{2}\times 1.$

$= \frac{1}{3} - \frac{3}{2} = \frac{2-9}{6} = \frac{-7}{6}$.

72. If A + B = 90°, then prove that:

$$\sqrt{\frac{\tan A \tan B + \tan A \cot B}{\sin A \sec B} - \frac{\sin^2 B}{\cos^2 A}} = \tan A.$$

2011/2012/2013/2014/2015/2016 [1 mark]

Given A + B = 90°, B = 90° - A

tan B = tan (90° - A) = cot A,

cot B = cot (90° - A) = tan A,

sin B = sin (90° - A) = cos A,

sec B = sec (90° - A) = cosec A,

$$\text{L.H.S} = \sqrt{\frac{\tan A . \cot A + \tan A . \tan A}{\sin A . \text{cosec } A} - \frac{\cos^2 A}{\cos^2 A}}$$

$$= \sqrt{\frac{1+\tan^2 A}{1} - 1} \qquad [\cot A = \frac{1}{\tan A} \text{and} \sin A = \frac{1}{\text{cosec } A}]$$

$= \sqrt{\tan^2 A}$ = tan A = R.H.S Hence proved.

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73. Evaluate: $(\sin^2 15° + \sin^2 75°) + \sqrt{3}(\tan 13° . \tan 60° . \tan 27° . \cot 20° . \cot 70° . \tan 77° . \tan 63°)$

2015/2016[2 mark]

We have:

$(\sin^2 15° + \sin^2 75°) + \sqrt{3}(\tan 13° . \tan 60° . \tan 27° . \cot 20° . \cot 70° . \tan 77° . \tan 63°)$

$= (\sin^2 15° + \cos^2 15°) + \sqrt{3}(\tan 13° . \tan 60° . \tan 27° . \tan 70° . \cot 70° . \cot 13° . \cot 27°)$

$= (1) + \sqrt{3}(\tan 13° . \cot 13° . \tan 60° . \tan 27° . \cot 27° . \tan 70° . \cot 70°)$ [cot (90° - θ = tan θ]

$= 1 + \sqrt{3}(1 \times \sqrt{3} \times 1 \times 1) = 1+3 = 4.$ $\left[\ \sin^2\theta + \cos^2\theta = 1, \cot\theta = \frac{1}{\tan\theta}\right]$

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74. Evaluate: $\dfrac{3\tan 25° \tan 40° \tan 50° \tan 65° - \frac{1}{2}\tan^2 60°}{4(\cos^2 29° + \cos^2 61°)}$

2014/2015/2016[3 marks]

$$\frac{3 \tan 25° \tan 40° \tan 50° \tan 65° - \frac{1}{2}\tan^2 60°}{4(\cos^2 29° + \cos^2 61°)}$$

$$= \frac{3 \tan 25° \tan 65° \tan 40° \tan 50° - \frac{1}{2}\tan^2 60°}{4(\cos^2 29° + \cos^2 61°)}$$

$$= \frac{3\tan 25° \times \tan(90°-25°)\tan 40° \tan(90°-40°) - \frac{1}{2}\tan^2 60°}{4(\cos^2 29° + \cos^2(90°-29°))}$$

$$= \frac{3\tan 25° \times \cot 25° \tan 40° \cot 40° - \frac{1}{2} \times (\sqrt{3})^2}{4(\cos^2 29° + \sin^2 29°)}$$

$$= \frac{3 \times 1 \times 1 - \frac{3}{2}}{4 \times 1} = \frac{3 - \frac{3}{2}}{4} = \frac{6-3}{2 \times 4} = \frac{3}{8}.$$

---

75. Show that: $\frac{1}{\sec\theta - \tan\theta} - \frac{1}{\cos\theta} = \frac{1}{\cos\theta} - \frac{1}{\sec\theta + \tan\theta}$

2013/2015/2016[3 marks]

We have: $\frac{1}{\sec\theta - \tan\theta} - \frac{1}{\cos\theta} = \frac{1}{\cos\theta} - \frac{1}{\sec\theta + \tan\theta}$

$\rightarrow \frac{1}{\sec\theta - \tan\theta} + \frac{1}{\sec\theta + \tan\theta} = \frac{1}{\cos\theta} + \frac{1}{\cos\theta} = \frac{2}{\cos\theta}$

Now, L.H.S $= \frac{1}{\sec\theta - \tan\theta} + \frac{1}{\sec\theta + \tan\theta}$

$= \frac{\sec\theta + \tan\theta + \sec\theta - \tan\theta}{(\sec\theta - \tan\theta)(\sec\theta + \tan\theta)}$

$= \frac{2\sec\theta}{\sec^2\theta - \tan^2\theta} = \frac{2\sec\theta}{1} = 2\sec\theta.$

$= 2 \times \frac{1}{\cos\theta} = \frac{2}{\cos\theta}$ = R.H.S

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76. Prove that $(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 = 7 + \tan^2\theta + \cot^2\theta$

2010/2011/2012/ 2013/2016[3 marks]

LHS $= \sin^2\theta + \operatorname{cosec}^2\theta + 2\sin\theta\operatorname{cosec}\theta + \cos^2\theta + \sec^2\theta + 2\cos\theta\sec\theta$

$= \sin^2\theta + \cos^2\theta + \operatorname{cosec}^2\theta + \sec^2\theta + 4$

$= 1 + 1 + \cot^2\theta + \tan^2\theta + 4$

$= 7 + \cot^2\theta + \tan^2\theta$ = RHS. Hence, proved.

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77. Prove that $(\operatorname{cosec}\theta - \cot\theta)^2 = \frac{1 - \cos\theta}{1 + \cos\theta}$

2010/2011/2012/ 2013/2015/2016[3 marks]

LHS $= (\operatorname{cosec}\theta - \cot\theta)^2$

$= \left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)^2 = \left(\frac{1-\cos\theta}{\sin\theta}\right)^2$

$= \frac{(1-\cos\theta)^2}{\sin^2\theta} = \frac{(1-\cos\theta)^2}{1-\cos^2\theta}$

$= \frac{(1-\cos\theta)^2}{(1-\cos\theta)(1+\cos\theta)} = \frac{1-\cos\theta}{1+\cos\theta}$ =RHS. Hence, proved.

78. If m sin θ + n cos θ = p and m cos θ – n sin θ = q, then prove that $m^2 + n^2 = p^2+q^2$.

2010/2011/2012/ 2013/2015/2016[3 marks]

We have: RHS = p^2+q^2

$=(m \sin \theta + n \cos \theta)^2 + (m \cos \theta - n \sin \theta)^2$

$=(m^2 \sin^2 \theta + n^2 \cos^2 \theta + 2mn \sin \theta \cos \theta) + (m^2 \cos^2 \theta + n^2 \sin^2 \theta - 2mn \sin \theta \cos \theta)$

$= m^2 (\sin^2 \theta + \cos^2 \theta) + n^2 (\sin^2 \theta + \cos^2 \theta)$

$= m^2 + n^2$ =LHS. Hence, proved.

---

79. If m = cos A – sin A and n = cos A + sin A, then show that

$$\frac{m}{n} - \frac{n}{m} = \frac{4\sin A\cos A}{\cos^2 A - \sin^2 A} = \frac{4}{\cot A - \tan A}$$

2014/2015/2016[4 marks]

We have: $\frac{m}{n} - \frac{n}{m} = \frac{m^2-n^2}{mn}$

$$= \frac{(\cos A-\sin A)^2-(\cos A+\sin A)^2}{(\cos A-\sin A)(\cos A+\sin A)}$$

$$= \frac{(\cos^2 A+\sin^2 A-2\sin A\cos A)-(\cos^2 A+\sin^2 A+2\sin A\cos A)}{\cos^2 A-\sin^2 A}$$

$= \frac{-4\sin A\cos A}{\cos^2 A-\sin^2 A}$, proved.

Now, $-\frac{4\sin A\cos A}{\cos^2 A-\sin^2 A} = \frac{\frac{4\sin A\cos A}{\sin A\cos A}}{\frac{\cos^2 A}{\sin A\cos A}-\frac{\sin^2 A}{\sin A\cos A}}$

$= \frac{-4}{\frac{\cos A}{\sin A}-\frac{\sin A}{\cos A}} = \frac{4}{\cot A-\tan A}$ = RHS. Hence, proved.

---

80. If x = tan A + sin A and y =tan A -sin A, prove that: $\left(\frac{x+y}{x-y}\right)^2 - \left(\frac{x+y}{2}\right)^2 = 1$.

2014/2015/2016[4 marks]

LHS $=\left(\frac{x+y}{x-y}\right)^2 - \left(\frac{x+y}{2}\right)^2$

$$= \left(\frac{\tan A+\sin A+\tan A-\sin A}{\tan A+\sin A-\tan A+\sin A}\right)^2 - \left(\frac{\tan A+\sin A+\tan A-\sin A}{2}\right)^2$$

$= \left(\frac{2\tan A}{2\sin A}\right)^2 - \left(\frac{2\tan A}{2}\right)^2 = \left(\frac{\sin A}{\cos A\sin A}\right)^2 - \tan^2 A.$

$= \sec^2 A - \tan^2 A = 1 + \tan^2 A - \tan^2 A = 1$ = RHS. Hence, proved.

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81. Prove that: $\frac{\text{cosec A}-\cot A}{\text{cosec A}+\cot A}+\frac{\text{cosec A}+\cot A}{\text{cosec A}-\cot A}=2(2\text{cosec}^2 A-1)=2\left(\frac{1+\cos^2 A}{1-\cos^2 A}\right).$

2014/2015/2016[4 marks]

LHS = $\frac{\text{cosecA}-\text{cotA}}{\text{cosecA}+\text{cotA}}+\frac{\text{cosecA}+\text{cotA}}{\text{cosecA}-\text{cotA}}$

$=\frac{(\text{cosecA}-\text{cotA})^2+(\text{cosecA}+\text{cotA})^2}{(\text{cosecA}-\text{cotA})(\text{cosecA}+\text{cotA})}$

$=\frac{\text{cosec}^2\text{A}+\cot^2\text{A}-2\text{cosecA cotA}+\text{cosec}^2\text{A}+\cot^2\text{A}+2\text{cosecAcotA}}{\text{cosec}^2\text{A}-\cot^2\text{A}}$

$=\frac{2\text{cosec}^2\text{A}+2\cot^2\text{A}}{1+\cot^2\text{A}-\cot^2\text{A}}=\frac{2\text{cosec}^2\text{A}+2(\text{cosec}^2\text{A}-1)}{1}$

$=\frac{4\text{cosec}^2\text{A}-2}{1}$ = 2(2cosec²A – 1). Hence, proved.

Now, $2(2\text{cosec}^2\text{ A}-1)$

$=2\left\{\frac{2}{\sin^2\text{A}}-1\right\}$

$=2\left\{\frac{2-\sin^2\text{A}}{\sin^2\text{A}}\right\}=2\left\{\frac{1+1-\sin^2\text{A}}{1-\cos^2\text{A}}\right\}=2\left\{\frac{1+\cos^2\text{A}}{1-\cos^2\text{A}}\right\}$. Hence, proved.

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82. Prove that: $\frac{\cot A+\text{cosec } A-1}{\cot A-\text{cosec } A+1}=\frac{1+\cos A}{\sin A}$

2011/2012/ 2013/2015/2016[4 marks]

LHS = $\frac{\text{cotA}+\text{cosecA}-(\text{cosec}^2\text{A}-\cot^2\text{A})}{\text{cotA}-\text{cosecA}+1}$ [∴ cosec² A – cot² A = 1]

$=\frac{(\text{cotA}+\text{cosecA})-(\text{cosecA}+\text{cotA})(\text{cosecA}-\text{cotA})}{\text{cotA}-\text{cosecA}+1}$

$=\frac{(\text{cotA}+\text{cosecA})(1-\text{cosecA}+\text{cotA})}{(\text{cotA}-\text{cosecA}+1)}$

= cot A + cosec A

$=\frac{\cos\text{A}}{\sin\text{A}}+\frac{1}{\sin\text{A}}=\frac{\cos\text{A}+1}{\sin\text{A}}$ =RHS. Hence, proved.

---

83. If tan θ + sin θ = m and tan θ – sin θ = n, show that $m^2-n^2=4\sqrt{mn}$.

2010/2011/ 2013/2015/2016[4 marks]

We have:

LHS = $m^2-n^2=(m+n)(m-n)$

= 2 tan θ. 2 sin θ = 4 tan θ sin θ

RHS = $4\sqrt{mn}=4\sqrt{(\tan\theta+\sin\theta)(\tan\theta-\sin\theta)}$

$=4\sqrt{\tan^2\theta-\sin^2\theta}=4\sqrt{\frac{\sin^2\theta}{\cos^2\theta}-\sin^2\theta}$

$= 4 \sin\theta \sqrt{\frac{1}{\cos^2\theta} - 1} = 4 \sin\theta \sqrt{\frac{1-\cos^2\theta}{\cos^2\theta}}$

$= 4 \sin\theta \sqrt{\frac{\sin^2\theta}{\cos^2\theta}} = 4 \sin\theta \tan\theta$

LHS = RHS → $m^2 - n^2 = 4\sqrt{mn}$.

---

84. If x = a sin θ and y = b tan θ, then prove that $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$.

2012/ 2013/2015/2016[4 marks]

$\frac{a^2}{x^2} - \frac{b^2}{y^2} = \frac{a^2}{a^2\sin^2\theta} - \frac{b^2}{b^2\tan^2\theta}$

$= \frac{1}{\sin^2\theta} - \frac{1}{\tan^2\theta} = \text{cosec}^2\theta - \cot^2\theta = 1.$

---

85. Prove that:

$\sqrt{\frac{\sec\theta-1}{\sec\theta+1}} + \sqrt{\frac{\sec\theta+1}{\sec\theta-1}} = 2\text{cosec}\,\theta.$

2010/2011/ 2012/ 2013/2016[4 marks]

LHS $= \sqrt{\frac{\sec\theta-1}{\sec\theta+1}} + \sqrt{\frac{\sec\theta+1}{\sec\theta-1}}$

$= \sqrt{\frac{(\sec\theta-1)}{(\sec\theta+1)} \times \frac{(\sec\theta-1)}{(\sec\theta-1)}} + \sqrt{\frac{\sec\theta+1}{\sec\theta-1} \times \frac{(\sec\theta-1)}{(\sec\theta+1)}}$

$= \sqrt{\frac{(\sec\theta-1)^2}{\sec^2\theta-1}} + \sqrt{\frac{(\sec\theta+1)^2}{\sec^2\theta-1}} = \frac{\sec\theta-1}{\sqrt{\sec^2\theta-1}} + \frac{\sec\theta+1}{\sqrt{\sec^2\theta-1}}$

$= \frac{(\sec\theta-1)+(\sec\theta+1)}{\sqrt{\sec^2\theta-1}}$

$= \frac{2\sec\theta}{\sqrt{\tan^2\theta}} = \frac{2\sec\theta}{\tan\theta} = \frac{2}{\cos\theta} \times \frac{\cos\theta}{\sin\theta}$

$= \frac{2}{\sin\theta} = 2\text{cosec}\,\theta$ = RHS. Hence, proved.